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PARAMETRIC AND SELF-EXCITED VIBRATIONS INDUCED BY FRICTION IN A SYSTEM WITH THREE DEGREES OF FREEDOM

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The paper presents the analysis of a nonlinear parametric system consisting of a rotor with rectangular cross-section placed in a rigid self-excited base. The parametric unstability zones have been identified on the basis of the method of expanding into a power series in relation to two perturbation parameters(one connected with parametric excitation, the other with friction coefficient). The influence of the changes of chosen parameters of the system on the size and shape of the unstability zones of the first order has been investigated.

Key Words: Rotor, Self-Excited and Parametric Vibraton, Perturbation Parameters

1. INTRODUCTION

Friction induced self-excited vibrations and parametric vibrations occur in many physical systems and have been in the focus of interest for a long time in many works concerning vibrations(Stoker, 1950 ; Minorsky, 1962 ; Hayashi, 1964 ; Cunningham, 1958). Both kinds of vibrations may be considered as sufficiently known. Howerver, when both excitations occur simultaneously in one system, the phenomenon is more complex (see for example, Alifov, Frolov ; 1985). On the other hand, this case occurs in technology, because e.g. in the combustion engine, in certain conditions self-excited vibrations of the piston and parametric vibrations of the crankshaft can occur. Both self-and parametrically excited vibrations together with forced vibrations are analysed in this paper. The parametric excitation and the exciting force come from the rotor with rectangular cross-section, which has in its middle a cylinder-like mass concentrated eccentrically on it. The rotor is fixed on a base placed on a belt moving at constant velocity. At a certain value of the belt velocity and the frequency of rotor turns, parametric and self-excited vibrations are created in addition to the forced vibrations.

As the parametric excitation μ and the friction coefficient ε are small in such a system they have been recognized as perturbation parameters. The methods with one perturbation parameter used to determine the limits of the stability-loss zones are widely described in the literature, and their extensive presentation is given by Malkin, 1956 ; Giacaglia, 1972 ; Iakubovic, Starzinsky, 1972. However, the analytical approach based on introducing of two independent perturbation parameters is rarely used in mechanics. This paper presents a general analytical technique for calculating the limits of stability in the system with self excited and parametric vibrations and develops author's earlier works (Awrejcewicz, 1986, 1989).

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2. THE ANALYSED SYSTEM AND EQUATIONS OF MOTION

The diagram of the analysed system is presented in Fig. 1. A weightless shaft with rectangular cross-section with a cylinder-like mass concentrated in its center is supported in the base placed on a belt moving at constant velocity V_o. The friction coefficient between the belt and the base depends on their relative velocity. The character of this dependence (Fig. 2) causes the creation of self-excited vibrations. The effect is described in the basic works on nonlinear vibrations. On the other hand, considering the non-identical cross-section of the rotor at some values of its rotational speed, parametric vibrations occur. The vibrations cause the changes of the normal force holding down the base to the belt in vertical direction, and hence they cause the changes of the friction force. It is assumed that the vibration of the rotor does not cause the tearing of the base off the belt.

The calculation model of the analysed system is presented in Fig. 3. The equations of motion of the system have the form :

$$\begin{aligned} m\ddot{x}_c &= -\xi_w k_\xi \cos\varphi - \eta_w k_\eta \sin\varphi \\ m\ddot{y}_c &= \xi_w k_\xi \sin\varphi - \eta_w k_\eta \cos\varphi + mg \\ I_{z''}\ddot{\varphi} &= -M_o + a(-\xi_w k_\xi \cos\varphi_o + \eta_w k_\eta \sin\varphi_o) \end{aligned} \qquad (1)$$

where

x_c, y_c : Coordinates of the centre of mas of the cylinder,

$I_{z''}$: Mass moment of inertia of a cylinder with mass m in relation to the z'' axis of the $O''x''y''z''$ system moving with translatory motion in relation to $Oxyz$

ξ_w, η_w : Coordinates of the point of puncture by the shaft in the coordinate system $o'\xi\eta$

$o'\xi\eta$: Coordinate system whose axes are parallel to the main, central inertia axes of the cross section of the shaft

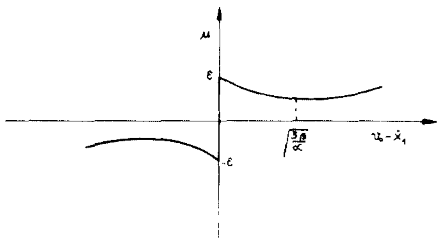

Fig. 1 Diagram of the analysed system

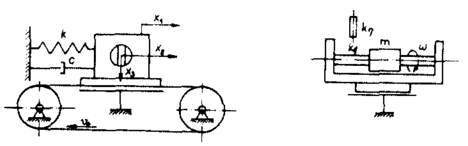

Fig. 2 Dependence of the friction coefficient on the relative velocity

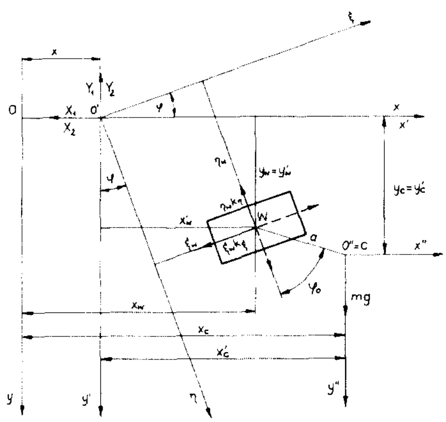

Fig. 3 Calculation model of the system

k_ξ, k_η : Shaft rigidities in the direction of the axes ξ and η

M_o : Driving torque reduced by all the resistance torques

a, φ_o : Parameters characterising the position of the centre of mass of the disk C in relation to the point of puncture by the shaft.

For the states near the steady ones the torque M_o is very small. Let

$$I_{z''} = m i_s^2 \tag{2}$$

where i_s is the inertia radius, then the third equation of the Eq. (1) will assume the form

$$\ddot{\varphi} = \frac{1}{m}\frac{a}{i_s^2}(-\xi_w k_\xi \cos\varphi_o + \eta_w k_\eta \sin\varphi_o) \tag{3}$$

As the eccentricity a and the shaft deflection ξ_w and η_w are small as compared to the inertia radius, then the following can be assumed :

$$\ddot{\varphi} = 0, \dot{\varphi} = \omega = \text{const}, \varphi = \omega t \tag{4}$$

The following geometric dependences result from the Fig. 3 :

$$\xi_w = (x_w - x)\cos\varphi - y_w \sin\varphi$$
$$\eta_w = (x_w - x)\sin\varphi + y_w \cos\varphi$$

$$y_c = y_w + a\cos(\varphi + \varphi_o)$$
$$x_c = x_w + a\sin(\varphi + \varphi_o) \tag{5}$$

where x_w, y_w are the coordinates of the point of puncture by the shaft W in the system Oxy.

In order to write down the equations of motion of the mass M it is necessary to determine the dynamic reactions on the shaft in its points of support. They are determined from the equations of equilibrium

$$X_1 + X_2 + \xi_w k_\xi \cos\omega t + \eta_w k_\eta \sin\omega t = 0$$
$$Y_1 + Y_2 - \xi_w k_\xi \sin\omega t + \eta_w k_\eta \cos\omega t = 0 \tag{6}$$

where X_1, Y_1 and X_2, Y_2 denote the support reactions on the left and right end of the shaft, respectively. The rotor reactions on the supports are then as follows

$$R_x = -X_1 - X_2$$
$$R_y = -Y_1 - Y_2 \tag{7}$$

The equation of motion of a body with mass M, on the assumption that $M_g + R_y > 0$, has the form

$$M\ddot{x} = -kx - c\dot{x} + R_x + (M_g + R_y)\mu(w), \tag{8}$$
$$w = v_o - \dot{x}$$

The dependence of the friction coefficient on the relative velocity w can be circumscribed with the polynomial

$$w = \varepsilon \operatorname{sgn} w - \alpha w + \beta w^3 \tag{9}$$

Finally, the equations of motion of the analysed system, after assuming that $x = x_1, x_w = x_2, y_w = x_3$, have the form :

$$\begin{aligned}
\ddot{x}_1 = & -x_1[\Omega^2 + \Omega^2{}_\xi + \Omega_\eta^2 + (\Omega_\xi^2 - \Omega_\eta^2)\cos 2\omega t] \\
& - H\dot{x}_1 - x_2[-(\Omega_\xi^2 + \Omega_\eta^2) + -(\Omega_\xi^2 - \Omega_\eta^2)\cos 2\omega t] \\
& - x_3(\Omega_\xi^2 - \Omega_\eta^2)\sin 2\omega t + \{g - x_2(\Omega_\xi^2 - \Omega_\eta^2)\sin 2\omega t + \\
& - x_3[-(\Omega_\xi^2 + \Omega_\eta^2) + (\Omega_\xi^2 - \Omega_\eta^2)\cos 2\omega t] \\
& + x_1(\Omega_\xi^2 - \Omega_\eta^2)\sin 2\omega t\} \cdot [\varepsilon \sin(v_o e \dot{x}_1) \\
& - \alpha(v_o - \dot{x}_1) + \beta(v_o - \dot{x}_1)^3] \\
\ddot{x}_2 = & \, x_1[\omega_\xi^2 + \omega_\eta^2 + (\omega_\xi^2 - \omega_\eta^2)\cos 2\omega t] \\
& - x_2[\omega_\xi^2 + \omega_\eta^2 + (\omega_\xi^2 - \omega_\eta^2)\cos 2\omega t] \\
& + x_3(\omega_\xi^2 - \omega_\eta^2)\sin 2\omega t + a\omega^2 \sin(\omega t + \varphi_o) \\
\ddot{x}_3 = & -x_1(\omega_\xi^2 - \omega_\eta^2)\sin 2\omega t + x_2(\omega_\xi^2 - \omega_\eta^2)\sin 2\omega t \\
& + x_3[-(\omega_\xi^2 + \omega_\eta^2) + (\omega_\xi^2 - \omega_\eta^2) \cdot \cos 2\omega t] \\
& + a\omega^2 \cos(\omega t + \varphi_o) + g
\end{aligned} \tag{10}$$

where : $\Omega^2 = \frac{k}{M}, \Omega_\xi^2 = \frac{k_\xi}{2M}, \Omega_\eta^2 = \frac{k_\eta}{2M}$,

$H = \frac{C}{M}, \omega_\xi^2 = \frac{k_s}{2m}, \omega_\eta^2 = \frac{k_\eta}{2m}$

3. TRANSFORMATION OF THE EQUATIONS OF MOTION TO THE MAIN COORDINATES

Let us introduce the following denotations

$$\Omega_1^2 = \Omega_\xi^2 + \Omega_\eta^2 \ ; \ \Omega_2^2 = \Omega_\xi^2 - \Omega_\eta^2 \ ;$$
$$\omega_1^2 = \omega_\xi^2 + \omega_\eta^2 \ ; \ \omega_2^2 = \omega_\xi^2 - \omega_\eta^2 \ ; \ \chi = \frac{\alpha}{\varepsilon} \ ; \ \rho = \frac{\beta}{\varepsilon} \ ; \ H = \mu H_1$$
$$; \ a\cos\varphi_o = \mu P \ ; \tag{11}$$
$$a\sin\varphi_o = \mu Q \ ; \ \mu G = g$$

where $\mu=\frac{\omega_2^2}{\omega_1^2}=\frac{\Omega_2^2}{\Omega_1^2}=\frac{k_\xi-k_\eta}{k_\xi+k_\eta}$ is the perturbation parameter.

After accounting for (11) in the equation system (10), it will assume the form

$$\begin{aligned}\ddot{x}_1&=-x_1\Omega^2-x_1\Omega_1^2(3+\mu\cos2\omega t)-\mu H_i\dot{x}_1\\&+x_2\Omega_1^2(1+\mu\cos2\omega t)+x_3\Omega_1^2\mu\sin2\omega t\\&+\varepsilon[g-x_2\Omega_1^2\mu\sin2\omega t+x_1\Omega_1^2\mu\sin2\omega t\\&+x_3\Omega_1^2(1-\mu\cos2\omega t)]\cdot[\mathrm{sgn}(v_o-\dot{x}_1)\\&-\chi(v_o-\dot{x}_1)+\rho(v_o-\dot{x}_1)^3];\\\ddot{x}_2&=-x_1\omega_1^2(1+\mu\cos2\omega t)-x_2\omega_1^2(1+\mu\cos2\omega t)\\&+x_3\omega_1^2\mu\sin2\omega t+\mu(P\sin\omega t+Q\cos\omega t)\omega^2\\\ddot{x}_3&=-x_1\omega_1^2\mu\sin2\omega t+x_2\omega_1^2\mu\sin2\omega t\\&-x_3\omega_1^2(1-\mu\cos2\omega t)+\mu(P\cos\omega t-Q\sin\omega t)\omega^2+\mu G\end{aligned}\tag{12}$$

When introducing $\mu=\varepsilon=0$ into the equation system (12), we obtain a homgeneous linear differential equation system

$$\begin{aligned}&\ddot{x}_1+x_1(\Omega^2+\Omega_1^2)-x_2\Omega_1^2=0\\&\ddot{x}_2+\omega_1^2(x_2-x_1)=0\\&\ddot{x}_3+\omega_1^2x_3=0\end{aligned}\tag{13}$$

When assuming the solution of (13) in the form $x_i=A_i\cos pt$, $i=1,2,3$, we find the following frequencies

$$\begin{aligned}&p_{1,2}^2=\frac{1}{2}\left[\Omega_2+\Omega_1^2+\omega_1^2\pm\sqrt{(\Omega^2+\Omega_1^2+\omega_1^2)^2-4\Omega^2\omega_1^2}\right]\\&p_3^2=\omega_1^2\end{aligned}\tag{14}$$

Let us introduce the main coordinates ξ_i, for which at $\mu=\varepsilon=0$ disjugation of the linear part of the first two equations of the system (12) will occur. Let us now multiply these equations by ξ_1 and ξ_2, respectively, and add the sides. The result will be

$$\begin{aligned}&\ddot{x}_1\xi_1+x_1(\Omega^2+\Omega_1^2)\xi_1-x_2\Omega_1^2\xi_1+\ddot{x}_2\xi_2+x_2\omega_1^2\xi_2\\&-x_1\omega_1^2\xi_2=\mu[-x_1\Omega_1^2\cos2\omega t\xi_1-H_1\dot{x}_1\xi_1\\&+\xi_1x_2\Omega_1^2\cos2\omega t-\xi_1x_3\Omega_1^2\sin2\omega t\\&+\xi_2x_1\omega_1^2\cos2\omega t-\xi_2x_2\omega_1^2\cos2\omega t\\&+\xi_2x_3\omega_1^2\sin2\omega t+\omega^2\xi_2(P\sin\omega t+Q\cos\omega t)]\\&+\varepsilon\xi_1[g-x_2\Omega_1^2\mu\sin2\omega t+x_3\Omega_1^2(1-\mu\cos2\omega t)\\&+x_1\Omega_1^2\mu\sin2\omega t]\cdot[\mathrm{sgn}(v_o-\dot{x}_1)-\chi(v_o-\dot{x}_1)\\&+\rho(v_o-\dot{x}_1)^3]\end{aligned}\tag{15}$$

By denoting

$$\begin{aligned}&(\Omega^2+\Omega_1^2)\xi_1-\omega_1^2\xi_2=\xi_1\Theta^2\\&-\Omega_1^2\xi_1+\omega_1^2\xi_2=\xi_2\Theta^2\end{aligned}\tag{16}$$

we find

$$\begin{aligned}&(\Omega^2+\Omega_1^2-\Theta^2)\xi_1-\omega_1^2\xi_2=0\\&-\Omega_1^2\xi_1+(\omega_1^2-\Theta^2)\xi_2=0\end{aligned}\tag{17}$$

In order for Eq. (17) to be fulfilled for ξ_1 and ξ_2 different from zero, the following dependence must occur

$$\begin{vmatrix}\Omega^2+\Omega_1^2-\Theta^2 & -\omega_1^2\\ -\Omega_1^2 & \omega_1^2-\Theta^2\end{vmatrix}=0\tag{18}$$

Thence

$$\Theta_1^2=p_1^2 \text{ and } \Theta_2^2=p_2^2$$

Let $\xi_1=\xi'_1$ and $\xi_2=\xi'_2$ be denoted for $\Theta_1=p_1$. From the second equation of the system (16) we find

$$\xi'_2=\gamma_1\xi'_1\tag{19}$$

where

$$\gamma_1=\frac{\Omega_1^2}{\omega_1^2-p_1^2}$$

Making use of the dependences (16) and (19), the Eq. (15) is transformed to the form

$$\begin{aligned}&\ddot{x}_1+x_1p_1^2+\ddot{x}_2\gamma_1+x_2p_1^2\gamma_1=\mu[x_1p_1^2\gamma_1\cos2\omega t\\&-H_1\dot{x}_1-p_1^2\gamma_1x_2\cos2\omega t+p_1^2\gamma_1x_3\sin2\omega t\\&+\gamma_1\omega^2(P\sin\omega t+Q\cos\omega t)]+\varepsilon[g-x_2\Omega_1^2\mu\sin2\omega t\\&+x_3\Omega_1^2(1-\mu\cos2\omega t)+x_1\Omega_1^2\mu\sin2\omega t]\\&\cdot[\mathrm{sgn}(v_o-\dot{x}_1)-x(v_o-\dot{x}_1)+\beta(v_o-\dot{x}_1)^3]\end{aligned}\tag{20}$$

Analogously, for $\Theta_2=P_2$ the following are denoted; $\xi_1=\xi''_1$ and $\xi_2=\xi''_2$, while

$$\xi''_2=\gamma_2\xi''_1\tag{21}$$

where:

$$\gamma_2=\frac{\Omega_1^2}{\omega_1^2-p_2^2}$$

Taking (16) and (21) into account in (15), the equation will assume the form

$$\begin{aligned}&\ddot{x}_1+x_1p_2^2+\ddot{x}_2\gamma_2+x_2\gamma_2p_2^2=\mu[x_1\gamma_2p_2^2\cos2\omega t\\&-H_1\dot{x}_1-x_2\gamma_2p_2^2\cos2\omega t+x_3\gamma_2p_2^2\sin2\omega t\\&+\gamma_2\omega^2(P\sin\omega t+Q\cos\omega t)]+\varepsilon[g-x_2\Omega_1^2\mu\sin2\omega t\\&+x_3\Omega_1^2(1-\mu\cos2\omega t)+x_1\Omega_1^2\sin2\omega t]\\&\cdot[\mathrm{sgn}(v_o-\dot{x}_1)-\chi(v_o-\dot{x}_1)\\&+\rho(v_o-\dot{x}_1)^3]\end{aligned}\tag{22}$$

Let us denote

$$\begin{aligned}&y_1=x_1+\gamma_1x_2\\&y_2=x_1+\gamma_2x_2\end{aligned}\tag{23}$$

The reverse dependences can be determined from the Eq. (23)

$$\begin{aligned}&x_1=\beta_1y_1-\beta_2y_2\\&x_2=\psi(y_1-y_2)\end{aligned}\tag{24}$$

where:

$$\beta_1=\frac{\gamma_2}{\gamma_2-\gamma_1},\beta_2=\frac{\gamma_1}{\gamma_2-\gamma_1},\psi=\frac{1}{\gamma_1-\gamma_2}$$

Let us additionally assume that $x_3=y_3$.

Taking (23) and (24) into account in (22), (20) and (12) we shall obtain the following differential equation system

$$\begin{aligned}&\ddot{y}_1+p_1^2y_1=\mu[p_1^2\gamma_1(\beta_1y_1-\beta_2y_2)\cos2\omega t\\&-H_1(\beta_1\dot{y}_1-\beta_2\dot{y}_2)-p_1^2\gamma_1\Psi\cdot(y_1-y_2)\cos2\omega t\\&+p_1^2\gamma_1y_3\sin2\omega t+\omega^2\gamma_1(P\sin\omega t+Q\cos\omega t)]\\&+\varepsilon[g-\mu\Omega_1^2\psi(y_1-y_2)\sin2\omega t+\mu\Omega_1^2(\beta_2y_1\\&-\beta_2y_2)\sin2\omega t+\Omega_1^2y_3(1-\mu\cos2\omega t)]\\&[\mathrm{sgn}(v_o-\beta_1\dot{y}_1+\beta_2\dot{y}_2)-\chi(v_o-\beta_2\dot{y}_1\end{aligned}$$

$$+\beta_2\dot{y}_2)+\rho(v_o-\beta_1\dot{y}_1+\beta_2\dot{y}_2)^3];$$
$$\ddot{y}_2+p_2^2y_2=\mu[p_2^2\gamma_2(\beta_1y_1-\beta_2y_2)\cos2\omega t$$
$$-H_1(\beta_1\dot{y}_1-\beta_2\dot{y}_2)+p_2^2\gamma_2\psi(y_1-y_2)\cos2\omega t$$
$$+p_2^2\gamma_2y_3\sin2\omega t+\omega^2\gamma_2(P\sin\omega t+Q\cos\omega t)]$$
$$+\varepsilon[g-\mu\Omega_1^2\psi(y_1-y_2)\sin2\omega t+\mu\Omega_1^2(\beta_1y_1$$
$$-\beta_2y_2)\sin2\omega t+\Omega_1^2y_3(1-\mu\cos2\omega t)][\mathrm{sgn}(v_o$$
$$-\beta_1\dot{y}_1+\beta_2\dot{y}_2)-\chi(v_o-\beta_1\dot{y}_1+\beta_2\dot{y}_2)$$
$$+\rho(v_o-\beta_1\dot{y}_1+\beta_2\dot{y}_2)^3] \quad (25)$$
$$\ddot{y}_3+p_3^2y_3=\mu[-p_3^2(\beta_1y_1-\beta_2y_2)\sin2\omega t$$
$$+p_3^2\psi(y_1-y_2)\sin2\omega t+p_3^2y_3\cos2\omega t$$
$$+\omega^2(P\cos wt-Q\sin\omega t)+G]$$

After introducing the dimensionless time $\tau=\omega t$, we obtain

$$\ddot{y}_1+\lambda_1^2y_1=\mu[\lambda_1^2\gamma_1(\varepsilon_1y_1-\varepsilon_2y_2)\cos2\tau$$
$$-\lambda_1\bar{H}_1(\beta_1y'_1-\beta_2y'_2)+\lambda_1^2\gamma_1y_3\sin2\tau$$
$$+\gamma_1(P\sin\tau+Q\cos\tau)]+\varepsilon[g+\mu\Omega_1^2(\varepsilon_1y_1-\varepsilon_2y_2)\sin2\tau$$
$$+\Omega_1^2y_3(1-\mu\cos2\tau)]\cdot\left[\frac{\lambda_1^2}{p_1^2}\mathrm{sgn}(v_o-\beta_1\omega y'_1-\beta_2\omega y'_2)\right.$$
$$+-\lambda_1\bar{\chi}_1(\lambda_1v'_o-\beta_1y'_1+\beta_2y'_2)$$
$$\left.+\omega\rho(\lambda_1v'_o-\beta_1y'_1+\beta_2y'_2)^3\right];$$
$$\ddot{y}_2+\lambda_2^2y_2=\mu[\lambda_2^2\gamma_2(\varepsilon_1y_1-\varepsilon_2y_2)\cos2\tau$$
$$-\lambda_2\bar{H}_2(\beta_1y'_1-\beta_2y'_2)+\gamma_2\lambda_2^2y_3\sin2\tau$$
$$+\gamma_2(P\sin\tau+Q\cos\tau)]+\varepsilon[g+\mu\Omega_1^2(\varepsilon_1y_1-\varepsilon_2y_2)\sin2\tau$$
$$+\Omega_1^2y_3(1-\mu\cos2\tau)]\cdot\left[\frac{\lambda_2^2}{p_2^2}\mathrm{sgn}(v_o-\beta_1\omega y'_1-\beta_2\omega y'_2)\right.$$
$$+\lambda_2\bar{\chi}_2(\lambda_2v''_o-\beta_1y'_1+\beta_2y'_2)$$
$$+\omega\rho(\lambda_2v''_o-\beta_1y'_1+\beta_2y'_2)^3 \quad (26)$$
$$\ddot{y}_3+\lambda_3^2y_3=\mu[-\lambda_3^2(\varepsilon_1y_1-\varepsilon_2y_2)\sin2\tau$$
$$+\lambda_3^2y_3\cos2\tau+P\cos\tau+Q\sin\tau+\lambda_3^2\bar{G}]$$

where:

$$yi=\frac{dy_i}{d\tau};\ \lambda_i^2=\frac{p_i^2}{\omega^2},\quad i=1,2,3$$
$$\varepsilon_k=\beta_k-\psi,k=1,2$$
$$\bar{\chi}_k=\frac{\chi}{p_k}$$
$$\bar{G}=\frac{G}{p_3^2};\ \bar{H}_1=\frac{H_1}{p_1};\ v'_o=\frac{v_o}{p_1};\ v''_o=\frac{v_o}{p_2};\ \bar{H}_2=\frac{H_2}{p_2}$$

4. ZONES OF UNSTABILITY OF THE FIRST ORDER

The procedure of solving the equation system (26) consists in assuming two perturbation parameters μ and ε connected with parameter excitation and friction, respectively.

The sought periodic solutions of $y_i(\tau)$ are presented in the form of a double power series:

$$y_i(\tau)=y_{0,0}^{(i)}+\mu y_{0,1}^{(i)}+\mu^2y_{0,2}^{(i)}+\cdots$$
$$+\varepsilon(y_{1,0}^{(i)}+\mu y_{1,1}^{(i)}+\mu^2y_{1,2}^{(i)}+\cdots)+ \quad (27)$$

where: $y_{k,1}^i,k,1=0,1,2,\cdots$ must fulfill the condition of periodicity. Periodic solutions are only possible for certain values of the parameters λ_i^2 presented in the form of analogous series:

$$\lambda_i^2=n^2+\mu\alpha_{0,1}+\mu^2\alpha_{0,2}+\cdots+\varepsilon(\alpha_{1,0}+\mu\alpha_{1,1}$$
$$+\mu^2\alpha_{2,2}+\cdots)+ \quad (28)$$

where: $\alpha_{k,1}$ $k,1=0,1,2\cdots$ are the unknown coefficients, which are determined from the condition of periodicity, avoiding in the solution terms unrestrictedly growing in time. For the resonance of the first order $n^2=1$ we shall determine the parametric unstability zones, for which the frequency of parameter modulation fulfills, consecutively, the dependences $\omega\cong p_1,\omega\cong p_2$,and $\omega\cong p_3$. In the series (27) and (28) for $\omega\cong p_1$ and $\omega\cong p_2$ we shall limit our considerations to the first powers of the small parameters μ and ε. On the other hand, for $\omega\cong p_3$ we shall limit ourselves in the calculations to the second approximation. In all the three cases we shall assume that $\mathrm{sgn}(v_o-\beta_1\omega y'_1-\beta_2\omega y'_2)=1$

Let

$$\lambda_2^2=\nu_{2,1}^2\lambda_1^2$$
$$\lambda_3^2=\nu_{3,1}^2\lambda_1^2 \quad (29)$$

where:

$$\nu_{2,1}=\frac{p_2}{p_1}$$
$$\nu_{3,1}=\frac{p_3}{p_1}$$

and let us assume that $\nu_{2,1}$ and $\nu_{3,1}$ are not integers.

Let us first consider the case $\omega\cong p_1$ assuming that

$$y_{0,0}^{(2)}(\tau)=y_{0,0}^{(3)}(\tau)=0 \quad (30)$$

The assumption is accounted for by a weak conjugation of the Eq. (26) for $\varepsilon\ll1$ and $\mu\ll1$. For $\mu=\varepsilon=0$ we shall obtain a disjugate system of three linear differential equations. For the resonance coordinate $y_1^{(r)}$, the magnitude of oscillation of the other two main coordinates should be of the order of the small parameters μ and ε.

Let us substitute the series (27) and (28) in the differential Eq. (26) taking into consideration the dependences (29) and (30) and the expansion

$$\lambda_1\cong1+\mu\frac{\alpha_{0,1}}{2}+\varepsilon\frac{\alpha_{1,0}}{2}+\cdots \quad (31)$$

After equating to zero the coefficients at the same powers ε and μ, we obtain a system of recurrent differential equations

$$y''^{(1)}_{0,0}+y^{(4)}_{0,0}=0$$
$$y''^{(1)}_{0,0}+y^{(1)}_{0,0}=-\alpha_{1,0}y^{(1)}_{0,0}+\frac{g}{p_1^2}--g\bar{\chi}_1v_o^1$$
$$+g\bar{\chi}_1\beta_1y'^{(1)}_{0,0}+g\omega\rho(v_o^1)^3-3g\omega\rho(v_o^1)^2\beta_1y'^{(1)}_{0,0}$$
$$+3g\omega\rho v_o^1\beta_1^2(y'^{(1)}_{0,0})^2+g\omega\rho\beta_1^3(y'^{(1)}_{0,0})^3;$$
$$y''^{(1)}_{0,0}+y^{(1)}_{0,0}=-\alpha_{0,1}y^{(1)}_{0,0}+\gamma_1\varepsilon_1y^{(1)}_{0,0}$$
$$\cos2\tau-\bar{H}_1\beta_1y'^{(1)}_{0,0}+\gamma_1P\sin\tau+\gamma_1Q\cos\tau;$$
$$y''^{(2)}_{1,0}+\nu_{2,1}^{(2)}y^{(2)}_{1,0}=g\cdot v_{2,1}^2/P_2^2gv_{2,1}^2\bar{\chi}_2v''_o$$
$$+g\nu_{2,1}\bar{\chi}_2\beta_1y'^{(1)}_{0,0}+g\omega\rho v_{2,1}^3(v''_o)^3$$
$$+3g\omega\rho v_{2,1}^2(v''_o)^2\beta_1y'^{(1)}_{0,0}$$
$$+3g\omega\rho\nu_{2,1}v''_o\beta_1^2(y'^{(1)}_{0,0})^2$$
$$+g\omega\rho\beta_1^3(y'^{(1)}_{0,0})^3; \quad (32)$$
$$y''^{(2)}_{0,1}+\nu_{2,1}^{(2)}y^{(2)}_{0,1}=\gamma_2\nu_{2,1}^2\varepsilon_1y^{(1)}_{0,0}\cos2\tau$$
$$-\nu_{2,1}\bar{H}_2\beta_1y'^{(1)}_{0,0}+\gamma_2P\sin\tau$$
$$+\gamma_2Q\cos\tau;y''^{(3)}_{1,0}+v_{3,1}^{(2)}y^{(3)}_{1,0}=0$$
$$y''^{(3)}_{0,1}+\nu_{3,1}^2y^{(3)}_{0,1}=-\nu_{3,1}^{(1)}\varepsilon_1y^{(1)}_{0,0}\sin2\tau$$
$$+P\cos\tau-Q\sin\tau+\nu_{3,1}^2\bar{G}$$

Assuming the solution of the first euqation of the system (32) in the form

$$y^{(1)}_{0,0}=a_1\cos\tau+b_1\sin\tau \quad (33)$$

we obtain the following from the second equation

$$
\begin{aligned}
y''^{(1)}_{1,0}+y^{(1)}_{1,0}=&\frac{p_1^2}{g}-g\bar{\chi}_1v'_o+g\omega\rho(v'_o)\\
&+\frac{3}{2}g\omega\rho v'_o\beta_1^2(a_1^2+b_1^2)+\cos\tau[-\alpha_{1,0}a_1\\
&+g\bar{\chi}\beta_1b_1-3g\omega\rho(v'_o)^2\beta_1b_1-\frac{3}{4}g\omega\rho\beta_1^3b_1^3\\
&+\frac{3}{4}g\omega\rho\beta_1^3b_1a_1^2+\sin\tau[-\alpha_{1,0}b_1\\
&-g\bar{\chi}_1\beta_1a_1+3g\omega\rho(v'_o)^2\beta_1a_1\\
&+\frac{3}{4}g\omega\rho\beta_1^3b_1^2a_1+\frac{3}{4}g\omega\rho\beta_1^3a_1^3]\\
&+\frac{3}{2}g\omega\beta v'_o\beta_1^2(b_1^2-a_1^2)\cos2\tau\\
&+3g\omega\rho v'_o\beta_1^2a_1b_1\sin2\tau+\cos3\tau[-\frac{1}{4}g\omega\rho\beta_1^3b_1^3\\
&+\frac{3}{4}g\omega\rho\beta_1^3b_1a_1^2]\\
&+\sin3\tau(-\frac{1}{4}g\omega\rho\beta_1^3a_1^3+\frac{3}{4}g\omega\rho\beta_1^3b_1^2a_1)
\end{aligned}
\tag{34}
$$

From the condition of periodicity we obtain two algebraic equations

$$
\begin{aligned}
&-\alpha_{1,0}a_1+(g\bar{\chi}_1\beta_1-3g\omega\rho(v'_o)^2\beta_1-\frac{3}{4}g\omega\rho\beta_1^3A_1^2)b_1=0\\
&-(g\bar{\chi}_1\beta_1-3g\omega\rho(v'_o)^2\beta_1-\frac{3}{4})g\omega\rho\beta_1^3A_1^2)a_1-\alpha_{1,0}b_1=0
\end{aligned}
\tag{35}
$$

where

$$A_1^2=a_1^2+b_1^2$$

For the non-zero a_1 and b_1 the following relation must occur

$$
\begin{vmatrix}
-\alpha_{1,0} & g\bar{\chi}_1(\beta_1-3g\omega\rho(v'_0)^2\beta_1-\frac{3}{4}g\omega\rho\beta_1^3A_1^2\\
-(g\bar{\chi}_1\beta_1-3g\omega\rho(v'_o)^2\beta_1-\frac{3}{4}g\omega\rho\beta_1^3A_1^2 & -\alpha_{1,0}
\end{vmatrix}=0
\tag{36}
$$

Thence

$$\alpha_{1,0}^2+(g\bar{\chi}_1\beta_1-3g\omega\rho(v'_o)^2\beta_1-\frac{3}{4}g\omega\rho\beta_1^3A_1^2)^2=0 \tag{37}$$

The only real solution of (37) is

$$
\begin{aligned}
&\alpha_{1,0}=0\\
&A_1^2=\frac{\bar{\chi}-3\omega\rho(v'_o)^2}{\frac{3}{4}\omega\rho\beta_1^2}
\end{aligned}
\tag{38}
$$

The following function is the solution of (34)

$$
\begin{aligned}
y^{(1)}_{1,0}=&\frac{g}{p_1^2}-g\bar{\chi}_1v'_o+g\omega\rho(v'_o)^3+\frac{3}{2}g\omega\rho v'_o\beta_1^2A_1^2\\
&+\frac{1}{2}g\omega\rho v'_o\beta_1^2(b_1^2-a_1^2)\cos2\tau\\
&+g\omega\rho v'_o\beta_1^2a_1b_1\sin2\tau+\Big(\frac{1}{32}g\omega\rho\beta_1^3b^3\\
&+\frac{3}{32}g\omega\rho\beta_1^3b_1a_1^2\Big)\cdot\cos3\tau+\Big(\frac{1}{32}g\omega\rho\\
&\beta_1^3a_1^3-\frac{3}{32}g\omega\rho\beta_1^3b_1^2a_1\Big)\sin3\tau
\end{aligned}
\tag{39}
$$

The solution omits the general integral of the homogeneous equation by associating it to $y^{(1)}_{0,0}$.

When substituting (33) in the fourth and sixth equation of the equation system (32), after transformations, we obtain

$$
\begin{aligned}
y''^{(2)}_{1,0}+\nu_{2,1}^2y^{(2)}_{1,0}=&\nu_{2,1}^2\frac{g}{p_2^2}-v_{2,1}^2g\bar{\chi}_2v''_o\\
&+g\omega\rho_{2,1}^3(v''_o)^3+\frac{3}{2}\nu_{2,1}g\omega\rho v''_o\beta_1^2A_1^2\\
&+\cos\tau\Big[\nu_{2,1}g\bar{\chi}_2\beta_1b_1-3g\omega\rho\nu_{2,1}^2\\
&(v''_o)^2\beta_1b_1+\frac{3}{4}g\omega\rho\beta_1^3A_1^2b_1\Big]+\sin\tau\Big[-\nu_{2,1}g\bar{\chi}_2\beta_1a_1\\
&+3g\omega\rho\nu_{2,1}^2(v''_o)^2\beta_1a_1\\
&+\frac{3}{4}g\omega\rho\beta_1^3A_1^2a_1\Big]+\frac{3}{2}\nu_{2,1}g\omega\rho v''_o\\
&\beta_1^2(b_1^2-a_1^2)\cos2\tau+3\nu_{2,1}g\omega\rho v''_o\beta_1^2a_1b_1\\
&\sin2\tau+\Big(-\frac{1}{4}g\omega\rho\beta_1^3b_1^3+\frac{1}{2}g\omega\rho\beta_1^3b_1a_1^2\Big)\\
&\cos3\tau+\Big(-\frac{1}{4}g\omega\rho\beta_1^3a_1^3+\frac{3}{4}g\omega\rho\beta_1^3b_1^2a_1\Big)\sin\tau
\end{aligned}
\tag{40}
$$

$$y''^{(3)}_{1,0}+\nu_{3,1}^2y^{(3)}_{1,0}=0 \tag{41}$$

The following functions are the paraticular solutions of the above equations:

$$
\begin{aligned}
y^{(2)}_{1,0}=&\frac{g}{p_2^2}\bar{\chi}_2v''_o+g\omega\rho\nu_{2,1}(v''_o)^3\\
&+\frac{3}{2\nu_{2,1}}g\omega\rho v''_o\beta_1^2A_1^2+\frac{1}{\nu_{2,1}^2-1}\\
&[\nu_{2,1}g\bar{\chi}_2\beta_1-3g\omega\rho\nu_{2,1}^2(v''_o)^2\beta_1\\
&-\frac{3}{4}g\omega\rho\beta_1^3A_1^2]b_1\cos\tau+\frac{1}{\nu_{2,1}^2-1}\\
&[-\nu_{2,1}g\bar{\chi}_2\beta_1a_1+3g\omega\rho\nu_{2,1}^2(v''_o)^2\\
&\beta_1a_1+\frac{3}{4}g\omega\rho\beta_1^3A_1^2a_1]\sin\tau+\frac{3\nu_{2,1}}{2(\nu_{2,1}^2-4)}\\
&g\omega\rho v''_o\beta_1^2a_1b_1\sin2\tau+\frac{1}{\nu_{2,1}^2-g}\\
&\Big(-\frac{1}{4}g\omega\rho\beta_1^3a_1^3+\frac{3}{4}g\omega\rho\beta_1^3b_1a_1^2\Big)\cos3\tau\\
&\frac{1}{v_{2,1}-g}(-\frac{1}{4}g\omega\rho\beta_1^3a_1^3+\frac{3}{4}g\omega\rho\beta_1^3b_1^2a_1)\sin3\tau
\end{aligned}
\tag{42}
$$

and

$$y^3_{1,0}=0 \tag{43}$$

By means of substituting (39) in the third equation of the system (32), we shall obtain

$$
\begin{aligned}
y''^1_{0,1}+y^{(1)}_{0,1}=&(-\alpha_{0,1}a_1+\frac{1}{2}\gamma_1\varepsilon_1a_1-\bar{H}_1\beta_1b_1+\gamma_1Q)\\
&\cos\tau+(-\alpha_{0,1}b_1-\frac{1}{2}\gamma_1\varepsilon_1b_1+\bar{H}_1\beta_1a_1+\gamma_1P)\\
&\sin\tau+\frac{1}{2}\gamma_1\varepsilon_1a_1\cos3\tau+\frac{1}{2}\gamma_1\varepsilon_1b_1\sin3\tau
\end{aligned}
\tag{44}
$$

We shall avoid terms unrestrictedly growing in time in its solution if the following equations are fulfilled:

$$
\begin{aligned}
&(\alpha_{0,1}-\frac{1}{2}\gamma_1\varepsilon_1)a_1+\bar{H}_1\beta_1b_1=\gamma_1Q\\
&-\bar{H}_1\beta_1a_1+(\alpha_{0,1}+\frac{1}{2}\gamma_1\varepsilon_1)b_1=\gamma_1P
\end{aligned}
\tag{45}
$$

For the case of $P=Q$, after transformations, we obtain the following from (45)

$$\alpha_{0,1}=$$
$$\pm\sqrt{\frac{P^2}{A_1^2}\gamma_1^2+\frac{1}{4}\gamma_1^2\varepsilon_1^2-\bar{H}_1^2\beta_1^2\pm\sqrt{(\frac{P}{A_1}\gamma_1)^4+\gamma_1^4\varepsilon_1^2\frac{P^2}{A_1^2}-2\bar{H}_1\beta_1\frac{P^2}{A_1^2}\gamma_1^3\varepsilon_1}} \tag{46}$$

The particular solution of (44) is

$$y_{0,1}^{(1)}=-\frac{1}{16}\gamma_1\varepsilon_1(a_1\cos3\tau+b_1\sin3\tau) \tag{47}$$

Taking (33) into consideration in the fifth and seventh equation of the system (32) we find the particular solutions

$$\begin{aligned}
y_{0,1}^{(2)}=&\frac{1}{\nu_{2,1}^2-1}\Big(\frac{1}{2}\nu_{2,1}^2\gamma_2\varepsilon_1a_1-v_{2,1}\bar{H}_2\beta_1b_1+\gamma_2Q\Big)\cos\tau\\
&+\frac{1}{\nu_{2,1}^2-1}\Big(-\frac{1}{2}\nu_{2,1}^2\gamma_2\varepsilon_1b_1\\
&+\nu_{2,1}\bar{H}_2\beta_1a_1+\gamma_2P)\sin\tau+\frac{\nu_{2,1}^2\gamma_2\varepsilon_1}{2(\nu_{2,1}^2-9)}\\
&a_1\cos3\tau+\frac{\nu_{2,1}^2\gamma_2\varepsilon_1}{2(\nu_{2,1}^2-9)}b_1\sin3\tau\\
y_{0,1}^{(3)}=&\bar{G}+\frac{1}{\nu_{3,1}^2-1}\Big(-\frac{1}{2}\nu_{3,1}^2\varepsilon_1b_1+P\Big)\cos\tau\\
&+\frac{1}{\nu_{3,1}^2-1}\Big(\frac{1}{2}\nu_{3,1}^2\varepsilon_1a_1+Q\Big)\sin\tau+\frac{\nu_{3,1}^2\varepsilon_1}{2(\nu_{3,1}^2-9)}\\
&b_1\cos3\tau+\frac{\nu_{3,1}^2\varepsilon_1}{2(\nu_{3,1}^2-9)}a_1\sin3\tau
\end{aligned} \tag{49}$$

We have thus determined the particular terms of the series (27) and (28), limiting the calcuations to the first approximation.

Let us now concentrate on the analysis of the case of $\omega\cong p_2$. The solutions will be sought, as has been done previously, in the form of the series (27) and (28) for $i=2$. From (28) we obtain

$$\lambda_2=1+\varepsilon\frac{\alpha_{1,0}}{2}+\mu\frac{\alpha_{0,1}}{2}+\cdots \tag{50}$$

Let us deonte that

$$\begin{aligned}
\lambda_1^2&=\nu_{1,2}^2\lambda_2^2\\
\lambda_3^2&=\nu_{3,2}^2\lambda_2^2
\end{aligned} \tag{51}$$

where $\nu_{1,2}=\frac{p_1}{p_2}$

$\nu_{3,2}=\frac{p_3}{p_2}$

and $\nu_{1,2}$ and $\nu_{3,2}$ are not integers.

Analogously to (30), we have

$$y_{0,0}^{(1)}(\tau)=y_{0,0}^{(3)}=0 \tag{52}$$

Substituting (27) in (26), with (50), (51) and (52) taken into account, after equating to zero the coefficients at the same powers of μ and ε, we obtain

$$\begin{aligned}
y''^{(1)}_{1,0}+\nu_{1,2}^2y_{1,0}^{(1)}=&\frac{g}{p_2^2}-g\bar{\chi}\nu_{1,2}^2v'_o-g\bar{\chi}\nu_{1,2}\\
&\beta_2y'^{(2)}_{0,0}+g\omega\rho\nu_{1,2}^3(v'_o)^3+3g\omega\rho\beta_2\nu_{1,2}^2\\
&(v'_o)^2y'^{(2)}_{0,0}+3g\omega\rho\nu_{1,2}\beta_2^2v'_o(y'^{(2)}_{0,0})^2\\
&+g\omega\rho\beta_2^3(y'^{(2)}_{0,0})^3\\
y''^{(1)}_{0,1}+\nu_{1,2}^2y_{0,1}^{(1)}=&-\gamma_1\nu_{1,2}^2\varepsilon_2y_{0,0}^{(2)}\cos2\tau+\nu_{1,2}\\
&\bar{H}_1\beta_2y'^{(2)}_{0,0}+\beta_1P\sin\tau+\gamma_1Q\cos\tau\ ;\\
y''^{(2)}_{0,0}+y_{0,0}^{(2)}=&0\ ;\\
y''^{(2)}_{1,0}+y_{1,0}^{(2)}=&-\alpha_{1,0}y_{0,0}^{(2)}+\frac{g}{p_2^2}-g\bar{\chi}_2v''_o\\
&-g\bar{\chi}_2\beta_2y'^{(2)}_{0,0}+g\omega\rho(v''_o)^3-3g\omega\rho(v''_o)^2\beta_2y'^{(2)}_{0,0}\\
&+3g\omega\rho v''_o\beta_2^2(y'^{(2)}_{0,0})^2+g\omega\rho\beta_2^3(y'^{(2)}_{0,0})^3\ ;\\
y''^{(2)}_{0,1}+y_{0,1}^{(2)}=&-\alpha_{0,1}y_{0,0}^{(2)}-\gamma_2\varepsilon_2y_{0,0}^{(2)}\cos2\tau\\
&+\bar{H}_2\beta_2y'^{(2)}_{0,0}+\gamma_2P\sin\tau+\gamma_2Q\cos\tau\\
y''^{(3)}_{1,0}+\nu_{3,2}^2y_{1,0}^{(3)}=&0\ ;\\
y''^{(3)}_{0,1}+\nu_{3,2}^2y_{0,1}^{(3)}=&\nu_{3,2}^2\varepsilon_2y_{0,0}^{(2)}\sin2\tau+P\cos\tau\\
&-Q\sin\tau+\nu_{3,2}^2\bar{G}
\end{aligned} \tag{53}$$

After substituting the following in the fourth equation of the system (53)

$$y_{0,0}^{(2)}=a_2\cos\tau+b_2\sin\tau \tag{54}$$

and using the trigonometric relations, we obtain

$$\begin{aligned}
y''^{(2)}_{1,0}+y_{1,0}^{(2)}=&\frac{g}{p_2^2}-g\bar{\chi}_2v''_o+g\omega\rho(v''_o)^3\\
&+\frac{3}{2}g\omega\rho(v''_o)_2^2(a_2^2+b_2^2)+(-\alpha_{1,0}a_2-g\bar{\chi}_2\beta_2b_2\\
&+3g\omega\rho(v''_o)^2\beta_1b_1+\frac{3}{4}g\omega\rho\beta_2^3b_2^3\\
&+\frac{3}{4}g\omega\rho\beta_2^3b_2a_2^2)\cos\tau+(-\alpha_{1,0}b_2\\
&+g\bar{\chi}_2\beta_2a_2-3g\omega\rho(v''_o)^2\beta_2a_2\\
&+\frac{3}{4}g\omega\rho\beta_2^3b_2^2a_2-\frac{3}{4}g\omega\rho\beta_2^3a_2^3)\sin\tau\\
&+\frac{3}{2}g\omega\rho v''_o\beta_2^2(b_2^2-a_2^2)\cos2\tau\\
&+3g\omega\rho v''_o\beta_2^2a_2b_2\sin_2\tau+\frac{1}{4}\\
&(b_2^2-3a_2^2)g\omega\rho\beta_2^3b_2\cos3\tau\\
&+\frac{1}{4}(a_2^2-3b_2^2)g\omega\rho\beta_2^3a_2\sin3\tau
\end{aligned} \tag{55}$$

From the condition of periodocity of the solution we get

$$\begin{aligned}
&-\alpha_{1,0}a_2+\Big(-g\bar{\chi}_2\beta_2+3g\omega\rho(v''_o)^2\beta_2\\
&\qquad+\frac{3}{4}g\omega\rho\beta_2^3A_2^2\Big)b_2=0\\
&\Big(g\bar{\chi}_2\beta_2-3g\omega\rho(v''_o)^2\beta_2-\frac{3}{4}g\omega\rho\beta_2^3A_2^2\Big)a_2\\
&\qquad-\alpha_{1,0}b_2=0
\end{aligned} \tag{56}$$

where $A_2^2=a_2^2=b_2^2$

For the non-zero a_2 and b_2 the main determinant of the equation system (56) must equal zero. From this condition we obtain

$$\begin{aligned}
\alpha_{1,0}&=0\\
A_2^2&=\frac{\bar{\chi}_2-3\omega\rho(v''_o)^2}{\frac{3}{4}\omega\rho\beta_2^2}
\end{aligned} \tag{57}$$

The particular solution of the Eq. (55) is

$$\begin{aligned}
y_{1,0}^{(2)}=&\frac{g}{p_2^2}-g\bar{\chi}_2v''_o+g\omega\rho(v''_o)^3+\frac{3}{2}g\omega\rho v''_o\beta_2^2A_2^2\\
&-\frac{1}{2}g\omega\rho v''_o\beta_2^2(b_2^2-a_2^2)\cdot\cos2\tau\\
&+g\omega\rho v''_o\beta_2^2a_2b_2\sin2\tau+\frac{1}{32}(3a_2^2-b_2^2)\,g\omega\rho\beta_2^3b_2
\end{aligned}$$

$$\cos 3\tau+\frac{1}{32}(3b_2^2-a_2^2)g\omega\rho\beta_2^3 a_2\sin 3\tau \quad (58)$$

Making use of (54) in the first and sixth equation of the system (53) we obtain their particular integrals

$$y_{1,0}^{(1)}=\frac{g}{p_1^2}-g\bar{\chi}_1 v'_o-g\omega\rho v_{1,2}(v'_o)^3$$
$$+\frac{3}{2\nu_{1,2}}g\omega\rho v'_o\beta_2^2A_2^2+\frac{1}{\nu_{1,2}^2-1}$$
$$-\left(-g\bar{\chi}_1\nu_{1,2}\beta_2+3g\omega\rho\beta_2\nu_{1,2}^2(v'_o)^2\right.$$
$$\left.+\frac{3}{4}g\omega\rho\beta_2^3A_2^2\right)b_2\cos\tau+\frac{1}{\nu_{1,2}^2-1}$$
$$\left(g\chi_1\nu_{1,2}\beta_2-3g\omega\rho\beta_2\nu_{1,2}^2(v'_o)^2\right.$$
$$\left.+\frac{3}{4}g\omega\rho\beta_2^3A_2^2\right)a_2\sin\tau+\frac{3\nu_{1,2}}{2(\nu_{1,2}^2-1)}g\omega\rho\beta_2^2$$
$$v'_o(b_2^2-a_2^2)\cos 2\tau+\frac{3\nu_{1,2}}{(\nu_{1,2}^2-4)}g\omega\rho\beta_2^2v'_oa_2b_2$$
$$\sin 2\tau\cdot\frac{1}{4}(b_2^2-3a_2^2)\frac{g\omega\rho\beta_2^3}{\nu_{1,2}^2-9}$$
$$b_2\cos 3\tau+\frac{1}{4}(a_2^2-3b_2^2)\frac{g\omega\rho\beta_2^3}{\nu_{1,2}^2-9}a_2\sin 3\tau \quad (59)$$

and

$$y_{1,0}^{(3)}=0 \quad (60)$$

The substitution of (54) in the fifth equation of the system (53) gives

$$y_{0,1}^{(2)}+y_{0,1}^{(2)}=\left(-\alpha_{0,1}a_2-\frac{1}{2}\gamma_2\varepsilon_2a_2+\bar{H}_2\beta_2b_2+\gamma_2Q\right)$$
$$\cos\tau+\left(-\alpha_{0,1}b_2+\frac{1}{2}\gamma_2\varepsilon_2b_2-\bar{H}_2\beta_2a_2+\gamma_2P\right) \quad (61)$$
$$\sin\tau+\frac{1}{2}\gamma_2\varepsilon_2a_2\cos 3\tau-\frac{1}{2}\gamma_2\varepsilon_2b_2\sin 3\tau$$

The following is obtained from the condition of periodicity after transformations and after assuming that $P=Q$

$$\alpha_{0,1}^4+2\alpha_{0,1}^2\left(\bar{H}_2^2\beta_2^2-\frac{1}{4}\gamma_2^2\varepsilon_2^2-\frac{P^2}{A_2^2}\gamma_2^2\right)$$
$$+\left(\bar{H}_2^2\beta_2^2-\frac{1}{4}\gamma_2^2\varepsilon_2^2)^2+\frac{1}{2}\gamma_2^4\varepsilon_2^2\frac{P^2}{A_2^2}\right.$$
$$-2\gamma_2^2\beta_2\bar{H}_2\frac{p^2}{A_2^2}(\bar{H}_2\beta_2-\gamma_2\varepsilon_2)=0 \quad (62)$$

Thence

$$\alpha_{0,1}=$$
$$\pm\sqrt{\frac{P^2}{A_2^2}\gamma_2^2+\frac{1}{4}\gamma_2^2\varepsilon_2^2-H_2^2\beta_2^2\pm\sqrt{\left(\frac{P}{A_2}\gamma_2\right)^4+\gamma_2^4\varepsilon_2^2\frac{P^2}{A_2^2}-2\bar{H}_2\beta_2\frac{P^2}{A_2^2}\gamma_2^3\varepsilon_2}} \quad (63)$$

The particular integer of the Eq. (61) is the following

$$y_{0,1}^{(2)}=\frac{\gamma_2\varepsilon_2}{16}(a_2\cos 3\tau+b_2\sin 3\tau) \quad (64)$$

On the other hand, after substituting (54) in the second and seventh equation of (53), we shall find the particular solutions

$$y_{0,1}^{(1)}=\frac{1}{\nu_{1,2}^2-1}\left(-\frac{1}{2}\nu_{1,2}\gamma_1\varepsilon_2a_2+\nu_{1,2}\bar{H}_1\beta_2b_2+\gamma_1Q\right)$$
$$\cos\tau+\frac{1}{2\nu_{1,2}^2-1}\left(\frac{1}{2}\nu_{1,2}^2\gamma_1\varepsilon_2b_2-\nu_{1,2}\right.$$
$$\left.\bar{H}_1\beta_2a_2+\gamma_1P\right)\sin\tau+-\frac{\nu_{1,2}^1\gamma_1\varepsilon_2}{2(\nu_{1,2}^2-9)}$$
$$a_2\cos 3\tau-\frac{\nu_{1,2}\gamma_1\varepsilon_2}{2(\nu_{1,2}^2-9)}b_2\sin 3\tau \quad (65)$$

$$y_{0,1}^{(3)}=\bar{G}+\frac{1}{\nu_{3,2}^2-1}\left(\frac{1}{2}\nu_{3,2}^2\varepsilon_2b_2+P\right)\cos\tau$$
$$+\frac{1}{\nu_{3,1}^2-1}\left(\frac{1}{2}\nu_{3,2}^2\varepsilon_2a_2-Q\right)\sin\tau$$
$$+\frac{\nu_{3,}^2\,2\varepsilon_2}{2(\nu_{3,2}^2-9)}b_2\cos 3\tau$$
$$-\frac{\nu_{3,2}^2\varepsilon_2}{2(\nu_{3,2}^2-9)}a_2\sin 3\tau \quad (66)$$

Finally, let us consider the case of $\omega\cong p_3$. Periodic solutions are possible for particular value of the parameter λ_3

$$\lambda_3\cong 1+\varepsilon\frac{\alpha_{1,0}}{2}+\mu\frac{\alpha_{0,1}}{2}+\varepsilon^2\frac{\alpha_{2,0}}{2}$$
$$+\mu^2\frac{\alpha_{0,2}}{2}+\varepsilon\mu\frac{\alpha_{1,1}}{2}+\cdots \quad (67)$$

Let us denote that

$$\lambda_1^2=\nu_{1,3}^2\lambda_3^2$$
$$\lambda_2^2=\nu_{2,3}^2\lambda_3^2 \quad (68)$$

where:

$$\nu_{1,3}=\frac{p_1}{p_3}$$
$$\nu_{2,3}=\frac{p_2}{p_3}$$

and $\nu_{1,3}$ and $\nu_{2,3}$ on assumption are not integers. Similarly to the previous considerations, assuming that

$$y_{0,0}^{(1)}(\tau)=y_{0,0}^{(2)}(\tau)=0 \quad (69)$$

we obtain the following recurrent differential equation system from the equation system (26)

$$y''^{(1)}_{0,0}+\nu_{1,3}^2y_{1,0}^{(1)}=(g+\Omega_1^2y_{0,0}^{(3)})\left[\frac{\nu_{1,3}^2}{p_1^2}\right.$$
$$\left.-\nu_{1,3}^2\bar{\chi}_1v'_o+\omega\rho\nu_{1,3}^3(v'_o)^3\right]$$
$$y''^{(1)}_{0,1}+\nu_{1,3}^2y_{0,1}^{(1)}=\gamma_1\nu_{1,3}^2y_{0,0}^{(3)}\sin 2\tau+\gamma_1(P\sin\tau+Q\cos\tau)$$
$$y''^{(1)}_{2,0}+\nu_{1,3}^2y_{2,0}^{(1)}=-\nu_{1,3}^2\alpha_{1,0}y_{1,0}^{(1)}+(g+\Omega_1^2y_{0,0}^{(3)})$$
$$\left[\frac{\alpha_{1,0}}{p_3^2}-\nu_{1,3}^2\bar{\chi}_1\alpha_{1,0}v'_o+-\nu_{1,3}\bar{\chi}_1(-\beta_1y'^{(1)}_{1,0}\right.$$
$$+\beta_2y'^{(2)}_{1,0}+\beta_2y'^{(2)}_{1,0}+\omega\rho\nu_{1,3}^2$$
$$\left.(v'_o)^2\left(\frac{\alpha_{1,0}}{2}\nu_{1,3}\nu'_0+\beta_1y'^{(2)}_{1,0}\right)\right]$$
$$+\Omega_1^2y_{1,0}^{(3)}\left[\frac{1}{p_3^2}-\nu_{1,3}^2\bar{\chi}_1v'_o+\omega\rho\nu_{1,3}^3(v'_o)^3\right]$$
$$y''^{(1)}_{0,2}+\nu_{1,3}^2y_{0,2}^{(1)}=-\nu_{1,3}^2\alpha_{0,1}y_{0,1}^{(1)}+\nu_{1,3}^2\gamma_1\left(\varepsilon_1y_{0,1}^{(1)}\right.$$
$$\left.-\varepsilon_2y_{0,1}^{(2)}\right)\cos 2\tau+\nu_{1,3}\bar{H}_1(\beta_1y_{0,1}^{(1)}$$
$$-\beta_2y'^{(2)}_{0,1}+\nu_{1,3}^2\gamma_1(\alpha_{0,1}y_{0,0}^{(3)}+y_{0,1}^{(3)})\sin 2\tau$$
$$y''^{(1)}_{1,1}+\nu_{1,3}^2y_{1,1}^{(1)}=-\nu_{1,3}^2(\alpha_{0,1}y_{1,0}^{(1)}+\alpha_{1,0}y_{0,1}^{(1)})$$
$$+\nu_{1,3}^2\gamma_1(\varepsilon_1y_{1,0}^{(1)}-\varepsilon_2y_{1,0}^{(2)})\cdot\cos 2\tau$$
$$-\nu_{1,3}\bar{H}_1(\beta_1y'^{(1)}_{1,0}-\beta_2y'^{(2)}_{1,0})+\nu_{1,3}^2\gamma_1(\alpha_{1,0}y_{0,0}^{(3)}+y_{1,0}^{(3)})\sin 2\tau$$
$$+(g+\Omega_1^2y_{0,0}^{(3)})\left[\frac{\alpha_{0,1}}{p_3^2}-\nu_{1,2}^2\bar{\chi}_1\alpha_{0,1}v'_o\right.$$

$$
\begin{aligned}
&-\nu_{1,3}\bar{\chi}_1(-\beta_1 y'^{(1)}_{0,1}+\beta_2 y^{(2)}_{0,1})+\omega\rho\nu_{1,3}^2\\
&(v'_o)^2\Big(\nu_{1,3}^2 v'_o\frac{\alpha_{0,1}}{2}-\beta_1 y'^{(1)}_{0,1}\\
&+\beta_2 y'^{(2)}_{0,1}+\Omega_1^2(y^{(3)}_{0,1}+y^{(3)}_{0,0}\cos 2\tau)\\
&\left(\frac{1}{p_3^2}-\nu_{1,3}^2\bar{\chi}_1 v'_o+\omega\rho\nu_{1,3}^3(v'_o)^3\right);
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
&y''^{(2)}_{1,0}+\nu_{2,3}^2 y^{(2)}_{1,0}=(g+\Omega_1^2 y^{(3)}_{0,0})\Big[\frac{\nu_{2,3}^2}{p_2^2}\\
&\quad-\nu_{2,3}^2\bar{\chi}_2 v''_o+\omega\rho\nu_{2,3}^2(v''_o)^3\Big];\\
&y''^{(2)}_{0,1}+\nu_{2,3}^2 y^{(2)}_{0,1}=\nu_{2,3}^2\gamma_2 y^{(3)}_{0,0}\sin 2\tau+\gamma_2(P\sin\tau+Q\cos\tau);\\
&y''^{(2)}_{2,0}+\nu_{2,3}^2 y^{(2)}_{2,0}=-\nu_{2,3}^2\alpha_{1,0}y^{(2)}_{1,0}+(g+\Omega_1^2 y^{(3)}_{0,0})\\
&\quad\Big[\frac{\alpha_{1,0}}{p_3^2}-\nu_{2,3}^2\bar{\chi}_2 v''_o\alpha_{1,0}+-\nu_{1,2}^2\bar{\chi}_2(-\beta_1 y'^{(1)}_{1,0}\\
&\quad+\beta_2 y'^{(2)}_{1,0})+\omega\rho\nu_{2,3}^2(v''_o)^2\\
&\quad\left(\frac{\alpha_{1,0}}{2}\nu_{2,3}v''_o+-\beta_1 y'^{(1)}_{1,0}+\beta_2 y'^{(2)}_{1,0}\right)\Big]\\
&\quad+\Omega_1^2 y^{(3)}_{1,0}\Big[\frac{1}{p_3^2}-\nu_{2,3}^2\bar{\chi}_2 v''_o+\omega\rho\nu_{2,3}^2(v''_o)^3\Big]\\
&y''^{(2)}_{0,2}+\nu_{2,3}^2 y^{(2)}_{0,2}=-\nu_{2,3}^2\alpha_{0,1}y^{(2)}_{0,1}+\nu_{2,3}^2\gamma_2\\
&\quad(\varepsilon_1 y^{(1)}_{0,1}-\varepsilon_2 y^{(2)}_{0,1})\cos 2\tau+\nu_{2,3}\bar{H}_2(\beta_1 y'^{(1)}_{1,0}\\
&\quad-\beta_2 y'^{(2)}_{0,1})+\nu_{2,3}^2\gamma_2(\alpha_{0,1}y^{(3)}_{0,0}+y^{(3)}_{0,1})\sin 2\tau\\
&y''^{(2)}_{1,1}+\nu_{2,3}^2 y^{(2)}_{1,1}=-\nu_{2,3}^2(\alpha_{0,1}y^{(2)}_{1,0}\\
&\quad+\alpha_{1,0}y^{(2)}_{0,1})+\nu_{2,3}^2\gamma_2(\varepsilon_1 y^{(1)}_{1,0}+\varepsilon_2 y^{(2)}_{1,0})\\
&\quad\cos 2\tau-\nu_{2,3}\bar{H}_2(\beta_1 y'^{(1)}_{1,0}-\beta_2 y^{(2)}_{1,0})\\
&\quad+\nu_{2,3}^2\gamma_2(\alpha_{1,0}y^{(3)}_{0,0}+y^{(3)}_{1,0})\sin 2\tau\\
&\quad+(g+\Omega_1^2 y^{(3)}_{0,0})\cdot\Big[\frac{\alpha_{0,1}}{p_3^2}\\
&\quad-\nu_{2,3}^2\bar{\chi}_2\alpha_{0,1}v''_o-\nu_{2,3}\bar{\chi}_2\\
&\quad(-\beta_1 y'^{(2)}_{0,1}+\beta_2 y'^{(2)}_{0,1})+\omega\rho\nu_{2,3}^2(v''_o)^2\\
&\quad\left(\nu_{2,3}v''_o\frac{\alpha_{0,1}}{2}-\beta_1 y'^{(1)}_{0,1}+\beta_2 y'^{(2)}_{0,0}\right)\Big]\\
&\quad+\Omega_1^2(y^{(3)}_{0,1}+-y^{(3)}_{0,0}\cos 2\tau)\\
&\quad\Big[\frac{1}{p_3^2}-\nu_{2,3}^2\bar{\chi}_2 v''_o+\omega\rho\nu_{2,3}^2(\nu''_0)^3\Big]\\
&y''^{(3)}_{0,0}+y^{(3)}_{0,0}=0\\
&y''^{(3)}_{1,3}+y^{(3)}_{1,0}=-\alpha_{1,0}y^{(3)}_{0,0}\\
&y''^{(3)}_{0,1}+y^{(3)}_{0,1}=-\alpha_{0,1}y^{(3)}_{0,0}+y^{(3)}_{0,0}\\
&\quad\cos 2\tau+P\cos\tau-Q\sin\tau+\bar{G}\\
&y''^{(3)}_{2,0}+y^{(3)}_{2,0}=-\alpha_{1,0}y^{(3)}_{1,0}-\alpha_{2,0}y^{(3)}_{0,0}\\
&y^{(3)}_{0,2}+y^{(3)}_{0,2}=-\alpha_{0,1}y^{(3)}_{0,1}-\alpha_{0,2}y^{(3)}_{0,0}\\
&\quad-\varepsilon_1 y^{(1)}_{0,1}\sin 2\tau+\varepsilon_2 y^{(2)}_{0,1}\sin 2\tau\\
&\quad+\alpha_{0,1}y^{(3)}_{0,0}\cos 2\tau+y^{(3)}_{0,1}\\
&\quad\cos 2\tau+\alpha_{0,1}\bar{G}\\
&y''^{(3)}_{1,1}+y^{(3)}_{1,1}=-\alpha_{1,1}s^{(3)}_{0,0}-\alpha_{1,0}y^{(3)}_{0,1}\\
&\quad-\alpha_{0,1}y^{(3)}_{1,0}-\varepsilon_1 y^{(3)}_{1,0}-\varepsilon_1 y^{(1)}_{1,0}\sin 2\tau\\
&\quad+\alpha_{0,1}y^{(3)}_{0,0}\cos 2\tau+y^{(3)}_{0,1}\cos 2\tau+\alpha_{0,1}\bar{G}\\
&y''^{(3)}_{1,1}+y^{(3)}_{1,1}=-\alpha_{1,1}s^{(3)}_{0,0}-\alpha_{1,0}y^{(3)}_{0,1}\\
&\quad-\alpha_{0,1}y^{(3)}_{1,0}-\varepsilon_1 y^{(1)}_{1,0}\sin 2\tau+\varepsilon_2 y^{(2)}_{1,0}\\
&\quad\sin 2\tau+\alpha_{1,0}y^{(3)}_{0,0}\cos 2\tau+y^{(3)}_{1,0}\cos 2\tau\\
&\quad+\alpha_{1,0}\bar{G}
\end{aligned}
$$

After substituting

$$y^{(3)}_{0,0}=a_3\cos\tau+b_3\sin\tau \tag{71}$$

in the twelfth equation of the system (70) we get

$$y''^{(3)}_{1,0}+y^{(3)}_{1,0}=-\alpha_{1,0}(a_3\cos\tau+b_3\sin\tau) \tag{72}$$

For the non-zero a_3 and b_3 the following results from the codition of periodicity

$$\alpha_{1,0}=0 \tag{73}$$

Thence

$$y^{(3)}_{1,0}=0 \tag{74}$$

Making use of (71) in the first and sixth equation of the system (70), we shall obtain their particular solutions

$$
\begin{aligned}
y^{(1)}_{1,0}=&\left[\frac{g}{\nu_{1,3}^2}+\frac{\Omega_1^2}{\nu_{1,3}^2-1}(a_3\cos\tau+b_3\sin\tau)\right]\\
&\left[\frac{\nu_{1,3}^2}{p_1^2}-\nu_{1,3}^2\bar{\chi}_1 v_0+\omega\rho\nu_{1,3}^2(v'_o)^3\right]
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
y^{(2)}_{1,0}=&\left[\frac{g}{\nu_{2,2}^2}+\frac{\Omega_1^2}{\nu_{2,3}^2-1}(a_3\cos\tau+b_3\sin\tau)\right]\\
&\left[\frac{\nu_{2,3}^2}{p_2^2}-\nu_{2,3}^2\bar{\chi}_2 v''_o+\omega\rho\nu_{2,3}^2(v''_o)^3\right]
\end{aligned}
\tag{76}
$$

After substituting (71) in the thirteenth equation of the system (70), we have

$$
\begin{aligned}
y''^{3}_{0,1}+y^{(3)}_{0,1}=&\,\bar{G}+(-\alpha_{0,1}a_3+\frac{1}{2}a_3+P)\cos\tau\\
&+(-\alpha_{0,1}b_3-\frac{1}{2}b_3-Q)\sin\tau+\frac{1}{2}(a_3\cos 3\tau+b_3\sin 3\tau)
\end{aligned}
\tag{77}
$$

The condition of periodicity gives

$$
\begin{aligned}
\alpha^{(1)}_{0,1}&=\frac{1}{2}+\frac{P}{a_3}\\
\alpha^{(2)}_{0,1}&=-\frac{1}{2}-\frac{Q}{b_3}
\end{aligned}
\tag{78}
$$

The particular solution of this equation is the following:

$$y^{(3)}_{0,1}=\bar{G}-\frac{1}{16}(a_3\cos 3\tau+b_3\sin 3\tau) \tag{79}$$

When substituting (71) in the second and seventh equation of the system (70), we obtain their particular integrals:

$$
\begin{aligned}
y^{(1)}_{0,1}=&\frac{\gamma_1}{\nu_{1,3}^2-1}\left(\frac{\nu_{1,3}^2}{2}b_3+Q\right)\cos\tau\\
&+\frac{\gamma_1}{\nu_{1,3}^2-1}\left(\frac{\nu_{1,3}^2}{2}a_3+P\right)\sin\tau+\frac{\nu_{1,3}^2\gamma_1}{2(\nu_{1,3}^2-9)}\\
&(a_3\sin 3\tau-b_3\cos 3\tau)
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
y^{(2)}_{0,1}=&\frac{\gamma_2}{\nu_{2,3}^2-1}\left(\frac{\nu_{2,3}^2}{2}b_3+Q\right)\cos\tau\\
&+\frac{\gamma_2}{\nu_{2,3}^2-1}\left(\frac{\nu_{2,3}^2}{2}a_3+P\right)\sin\tau+\frac{\nu_{2,3}^2\gamma_2}{2(\nu_{2,3}^2-9)}\\
&(a_3\sin 3\tau-b_3\cos 3\tau)
\end{aligned}
\tag{81}
$$

After substituting (71) and (73) in the fourteenth equation of the system (70), we obtain the following from the condition of the existance of periodic solutions

$$\alpha_{2,0}=0 \tag{82}$$

Analogously, taking (71), (79), (80), (81) into account in the fifteenth equation of the system (70), we shall obtain equations which, after transformations, will assume the from

$$
\begin{aligned}
\alpha^{(1)}_{0,2}=&\frac{\gamma_1\varepsilon_1\nu_{1,3}^2}{4}\left(\frac{\nu_{1,3}^2-1}{2}+\frac{1}{\nu_{1,3}^2-9}\right)\\
&+\frac{\gamma_2\varepsilon_2\nu_{2,3}^2}{4}\cdot\left(\frac{1}{\nu_{2,3}^2-1}+\frac{1}{\nu_{2,3}^2-9}\right)\\
&+\frac{1}{2}\alpha^{(1)}_{0,1}-\frac{1}{32}-\frac{P}{a_3}\Big(\frac{\varepsilon_1\gamma_1}{2(\nu_{1,3}^2-1)}\\
&-\frac{\varepsilon_2\gamma_2}{2(\nu_{2,3}^2-1)}\Big)
\end{aligned}
\tag{83}
$$

$$a_{0,2}^{(2)} = -\frac{\gamma_1\varepsilon_1\nu_{1,3}^2}{4}\left(\frac{1}{\nu_{1,3}^2-1}+\frac{1}{\nu_{1,3}^2-9}\right) + \frac{\gamma_2\varepsilon_2\nu_{2,3}^2}{4}\cdot\left(\frac{1}{\nu_{2,3}^2-1}+\frac{1}{\nu_{2,3}^2-1}\right) - \frac{1}{2}a_{0,1}^{(2)} - \frac{1}{32} - \frac{Q}{b_3}\left(\frac{\varepsilon_1\gamma_1}{2(\nu_{1,3}^2-1)}+\frac{\varepsilon_2\gamma_2}{2(\nu_{2,3}^2-1)}\right) \quad (84)$$

The following algebraic equation system will be obtained from the condition of periodicity of the solutions of the equation system (70) after substituting (71), (74), (75), (76) and (79) in its sixteenth equation:

$$-a_{1,1}a_3 - \frac{\varepsilon_1 c_1 \Omega_1^2}{2(\nu_{1,3}^2-1)}b_3 + \frac{\varepsilon_2 c_2 \Omega_1^2}{2(\nu_{2,3}^2-1)}b_3 = 0$$
$$-a_{1,1}b_3 - \frac{\varepsilon_1 c_1 \Omega_1^2}{2(\nu_{1,3}^2-1)}a_3 + \frac{\varepsilon_2 c_2 \Omega_1^2}{2(\nu_{2,3}^2-1)}a_3 = 0 \quad (85)$$

where:

$$c_1 = \frac{\nu_{1,3}^2}{p_1^2} - \nu_{1,3}^2\bar{\chi}_1 v'_o + \omega\rho\nu_{1,3}^3(v'_o)^3$$
$$c_2 = \frac{\nu_{2,3}^2}{p_2^2} - \nu_{2,3}^2\bar{\chi}_2 v''_o + \omega\rho\nu_{2,3}^2(v''_o)^3$$

From the condition of a non-zero solution of the equation system (85) in relation to a_3 and b_3, we obtain

$$a_{1,1}^{(1,2)} = \pm\left[\frac{\Omega_1^2}{2}\left(\frac{\varepsilon_2 c_2}{\nu_{2,3}^2-1} - \frac{\varepsilon_1 c_1}{\nu_{1,3}^2-1}\right)\right] \quad (86)$$

The coefficients of the sought series (67) are determined by the expressions (73), (78), (83), (84) and (86).

5. CALCULATION EXAMPLES

The analytically obtained diagrams of parametric unstability zones are presented below in order to illustrate the influence of particular parameters of the system on their magnitude and position. The physical parameters of the system are given in the form in which they occur in the differential Eq.(12).

Figures 4, 5 present the influence of unbalance μP, damping (μH_1), and the shape of friction characteristic (α/β) on the magnitude of the parametric unstability zones for p_1 and p_2, for the following data: $\Omega^2=900s^{-2}$, $\Omega_1^2=480s^{-2}$, $\omega_1^2=4800s^{-2}$, $g=\mu G=9,81ms^{-2}$, $v_0=0,4ms^{-1}$, $\varepsilon=0,2$. On the basis of (14), $p_1=73,32s^{-1}$ $p_2=28,35s^{-1}$, and $p_3=69,28s^{-2}$ have been obtained. The adequate coefficients assume the form $\gamma_1=0,833$, $\gamma_2=0,12$, $\beta_1=0,126$, $\beta_2=-0,674$ $\varepsilon_1=1,176$, $\varepsilon_2=0,376$. The other quantities chavacterising the system have been marked in the figures (while $\alpha/\beta=\alpha/\rho$).

The parametric unstability zones presented in Figs. 4(for p_1) and 5(for p_2) expand with the increase of unbalance μP, while, depending on the value of the quotient α/β, this tendency can have different intensity. In the case of $\alpha/\beta=0,5m^2s^2$ the doubling of unbalance has caused the expansion of the unstability to double for zones for p_1 as well as for p_2. For $\alpha/\beta=1m^2s^{-2}$the increase a tripling of unbalance brings about a comparatively small expansion of the unstability zones for p_1, while for p_2 the expan sion is still almost doubled. In the case of large unbalance of the rotor, the changes of the quotient α/β do not influence the maguitude of the parametric unstability zones. The influence of damping on the magnitude

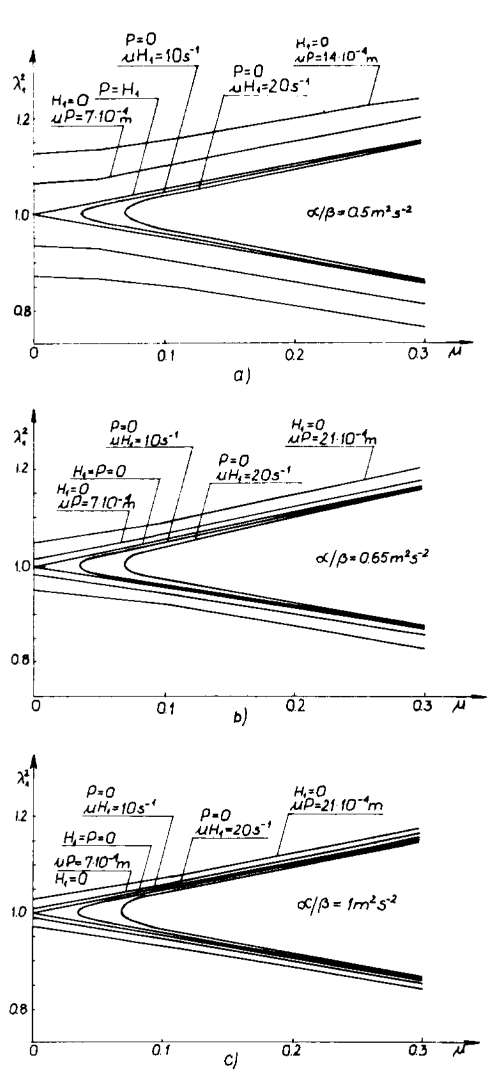

Fig. 4 Unstability zones for p_1

of the unstability zones corresponding to the frequencies p_1 and p_2 is also very different. Small damping ($\mu H_1=0,05s^{-1}$) causes considerable shift of the zone for p_2 in the direction of the growing value of modulation depth μ($\mu\geq 0,15$). In the case of double increase of damping the zone will not occur for $\mu\leq 0,3$.

The magnitude and position of the unstability zones for p_1 are not so sensitive to the changes of the damping coefficient. In the case of $\mu H_1=10s^{-1}$the proper zone of frequency p_1 exists for $\geq 0,034$. After a doubled increase of damping, when $\mu H_1=20s^{-1}$, the lower border of the occurence of the zone is shifled to the value of $\mu=0,07$.

The parametric unstability zones for p_3 are presented in Fig. 6. The magnitude of the zones depends on the initial conditions of the system's motion. The diagrams have been prepared on the assumption that $a_3=b_3=0,01m$ where $a_3=y_3(0)$, $b_3=y_3(0)$. The calculations, in the case of the resonance coordinate y_3, have been performed with an exactitude up to the second approximation, thence the inclination of the unstability zones in the direction of the growing values of the parameter λ_3^2. For the first approximation, the zones remain symmetrical in relation to the straight line $\lambda_3^2=1$. As in the cases considered aboves the increase of unbalance considerably expands the unstability zone. The changes of the value

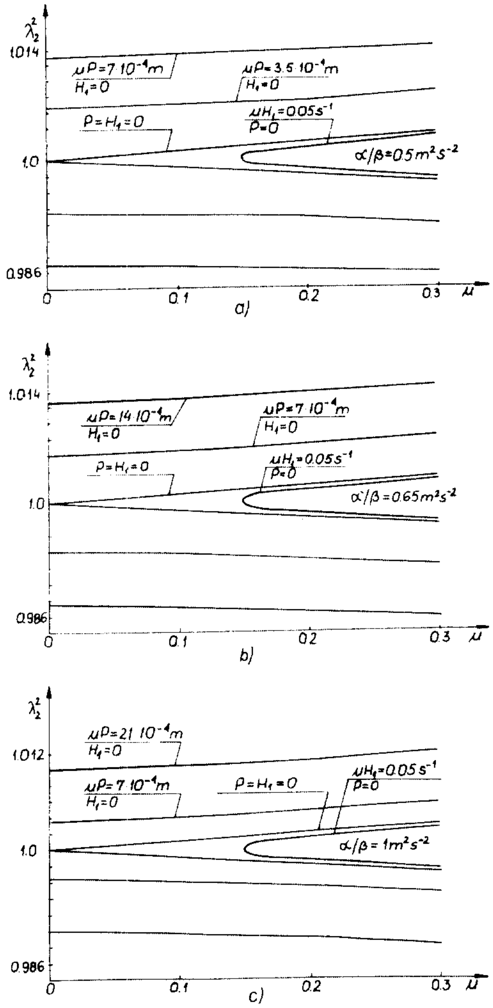

Fig. 5 Unstability zones for p_2

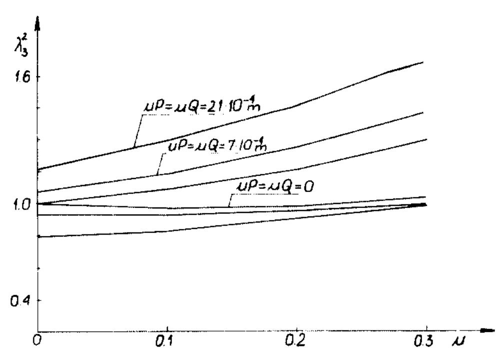

Fig. 6 Unstability zones for $p_3(\alpha/\beta=0,\ 5\text{m}^2\text{s}^{-2},\ \alpha/\beta=0.65\text{m}^2\text{s}^{-2},\ \alpha/\beta=1\text{m}^2\text{s}^{-2})$

of the quotient α/β and damping have a negligible influence on the magnitude of the zone. Fig. 7 presents the parametric unstability zones for various values of the parameters Ω^2, Ω_1^2 and ω_1^2. For the zones denoted by 1 we get $\Omega^2=14400\text{s}^{-2}$, $\Omega_1^2=1920\ \text{s}^{-2}$, $\omega_1^2=19200\text{s}^{-2}$; for the zones denoted by 2 we have $\Omega^2=3600\text{s}^{-2}$, $\Omega_1^2=480\text{s}^{-2}$, $\omega_1^2=4800\text{s}^{-2}$, and for the zones denoted by 3 : $\Omega^2=900\text{s}^{-2}$, $\Omega_1^2=120\text{s}^{-2}$, $\omega_1^2=1200\text{s}^{-2}$. In all the cases the magnitudes of the other parameters are as follows : $\mu H_1=10\text{s}^{-1}$, $\varepsilon=0.2$, $\alpha/\beta=0.5\text{m}^2\text{s}^{-2}$, $v_0=0.4\text{ms}^{-1}$,

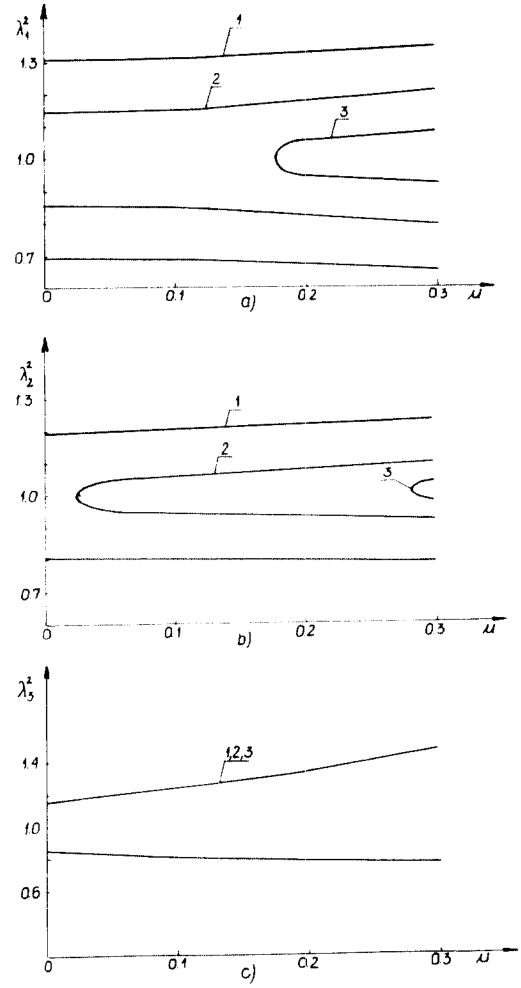

Fig. 7 Influence of the parameter changes on the unstability zones for : a/p_1 ; b/p_2 ; c/p_3

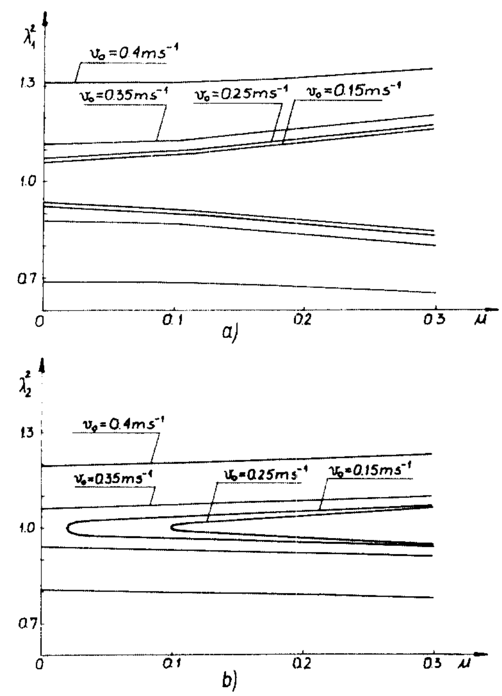

Fig. 8 Influence of the belt velocity changes v_0 on the unstability zones for : a/p_1 ; b/p_2

$\mu P = 0,0015$m.

As results from the Fig. 7(a),(b) the growth of the squares of frequencies $\Omega^2, \Omega_1^2, \omega_1^2$ (resulting from the increase of rigidity of the elastic elements in the system, or from the decrease of the values of masses) causes that the unstability zones for p_1 and p_2 expand. For example, the increase of the parameters Ω^2, Ω_1^2, and ω_1^2 by four times brings about an approximately doubled expansion of the zones. The unstability zone for p_3 is not influenced by the frequency changes in the system[Fig. 7(c)].

Figure 8(a),(b) presents the influence of the velocity changes of the belt v_0 on the magnitude of the unstability zones for p_1 and p_2. Calculations have been performed for the data denoted by 1, except for the velocity v_0, whose value has been changed. In each case the velocity increase of the belt causes the expansion of the unstability zones. For $v_0 < 0,3$ms^{-1} these changes ae less evident. The influence of the velocity changes v_0 on the parametric unstability zone for p_3 is practically negligible.

6. CONCLUDING REMARKS

The paper presents the analysis of a discrete mechanical system with three degrees of freedom, where self excited vibrations caused by friction, as well as parametric and forced vibrations occur. The system of ordinary differential equations governing the motion of the analysed system is nonlinear of the six order. The periodicity of the coefficients in the linear section of the equation of motion results from non-identical moments of inertia of the shaft cross section of the rotor constituting part of the analysed system. The nonlinearity is introduced into the equations of motion by friction between the belt and the rigid mass element where the rotor is placed. Moreover, it is increased by the normal reaction changes between the belt and the rotor base resulting from the rotor vibrations. External excitation in the form of a periodic function of time is also introduced into the system; the excitation is the effect of the unbalance of the rotor.

The analysis performed makes it possible to present the following conclusions:

(1) The method of seeking a solution as the power series of two perturbation parameters μ and ε used in the considerations makes it possible to investigate the single resonances of any order for the systems with weak nonlinearity and weakly modulated($\mu \ll 1$).

When performing calculations with an exactitude up to the second approximation, it turns out that the limits of unstability zones incline in the direction of the growing values of the parameter λ_3^2(Fig. 6). For the first approximation, the limits remain symmetrical in relation to the straight line $\lambda_3^2 = 1$.

(2) The parametic unstability zones for p_1 and p_2 expand with the increase of the rotor unbalance. Depending on the value of the quotient α/p, this tendency has different intensity. In the case of $\alpha/\beta = 0,5$ m^{2}s^{-2}, a double increase of inbalance has brought about a considerable expansion of the unstability zones, for p_1 as well as for p_2. For $\alpha/\beta = 1$ m^{2}s^{-2} the unbalance which causes a rather small expansion of the unstability zones for p_1, while for p_2 the expansion is still almost double. In the case of lack of the rotor unbalance the changes of the quotient α/β do not influence the magnitude of the parametric unstability zones. The influence of damping on the magnitude of the unstability zones corresponding to p_1 and p_2 is also very different. Minimum damping ($\mu H_1 = 0,05$ s^{-1}) causes considerable shift of the zone for p_2 in the direction of the growing values of modulation depth ($\mu H_1 = 0,15$ s^{-1}). The magnitude and position of the unstability zones for p_1 are not so sensitive to the damping coefficient changes. The regularities indicated here are the more clear, the greater is the dif-freence between the values of frequency p_1 and p_2 (ie. for $p_1 \gg p_2$). The increase of unbalance also produces a considerable expansion of the unstability zone for p_3; however the changes of the parameter and of damping have no essential influence on the magnitude of the zone.

The growth of the frequency squares Ω^2, Ω_1^2 and ω_1^2 causes the expansion of the unmstability zones for p_1 and p_2. The parametric unstability zone for the frequency p_3 is not sensitive to the frequency changes in the system.

In the case of the belt velocity increase v_0, the unstability zones for p_1 and p_2 are expanded. This property is noticeable within the range of great velocities($v_0 > 0,3$ms^{-1}). The influence of the velocity changes v_0 on the parametric unstability zone for p_3 is practically negligible.

(3) For the frequencies p_1 and p_2, the position of unstability zone limits does not depend in the first approximation on the initial conditions of the system's motion. The magnitude of the unstability zone limits for p_3 depends on them.

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